Week 7 (2) –More about Matching

1. More proofs of Hall's Marriage Theorem

First, let's recall the theorem. Let H be a bipartite graph with bipartition (B, G), with condition as follows:

Condition: every subset of k boys collectively fancy at least k girls $(*)$

Theorem: There exists a matching for all $b \in B$ if and only if the condition $(*)$ holds for all subset of$B$.

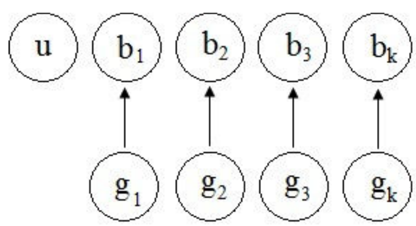
The proof for the sufficient direction is straightforward. If there exist k boys who fancy less than k girls, then the matching can't happen. It remains to proof the other direction.

Proofs:

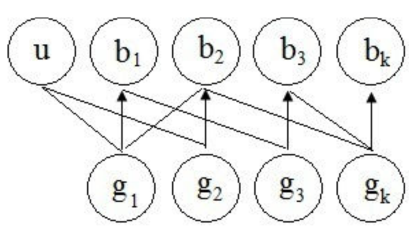
1. Berge's Theorem

We use contra positive for this proof.

Suppose such a matching doesn't exist. Choose a maximum matching M. There exist $u \in B$ not incident to an edge in M.



Let $Z \subset V(G)$ be the set of vertices reachable from u, and define $Y := Z \cap B$ and $X \coloneqq Z \cap G$. By Berge's theorem, there exists no augmenting path inside Z.



Since there is no augmenting path, any path starting at u must end at some $b_i \in Y$. So Z contains odd number of vertices, and we have $|N(Y)| = |X| = |Y| - 1 < |Y|$. Therefore, the condition $(*)$ is violated and the proof is complete.

2. Hall’s Original Version of the Theorem

We will state the original version of Hall’s theorem and show how they are equivalent. Let $\boldsymbol{S} = \{S_1, S_2, \dots, S_m\}$ be collection of subsets of a finite set S.

Definition: A transversal is a set of m distinct elements of S; one in each S_i.

Theorem:

S has a transversal $\Leftrightarrow$ union of any k subsets $S_1, S_2, \dots, S_m$ of S contain at least k distinct elements.

Let t_i be elements of S. Construct a bipartite graph with bipartition $t_1, t_2, \dots, t_n$ and $S_1, S_2, \dots, S_m$. Connect t_i to S_j if and only if $t_i \in S_j$.

- $t_1, t_2, \dots, t_n$ correspond to set of girls.
- $S_1, S_2, \dots, S_m$ correspond to the set of boys.
- Union of any k subsets contains at least k elements, thus connected to at least k elements of $t_1, t_2, \dots, t_n$, which corresponds to the condition $(*)$.
- There is a transversal means there is perfect matching.

so indeed, they are equivalent.

2. Kȍnig-Egervary Theorem

Definition:

Covering of a graph: Subset $K \subset V(G)$ such that every edge of G is incident to an element of K.

Minimal covering: A covering of G such that $|K| = \beta(G)$ is minimal.

Theorem:

Let G be a bipartite graph. Let $\alpha'(G)$ be the number of edges in maximum matching in G. Then, $\alpha'(G)$ is equal to $\beta(G)$.

Proof:

Let M be the $m \times n$ adjacency matrix of $G = ((G_1, G_2), E)$, with $|G_1| = m$ and $|G_2| = n$. Thus, we can get the adjacency matrix of the form

$$\begin{array}{cc} \begin{array}{cc} m-r & r \end{array} & \\ \begin{pmatrix} A & C \\ O & B \end{pmatrix} & \begin{array}{c} s \\ n-s \end{array} \end{array}$$

where $r + s = \beta(G)$.

$\alpha'(G)$ is the maximum number of 1’s that does not lie in the same row or column. Marriage condition holds, so A contains s 1’s, no two of which lie on the same row or column. Similarly for B. So $\alpha'(G) \geq r + s = \beta(G)$. But Clearly $\alpha'(G) \leq \beta(G)$.